Reference for Discrete Calculus and Functional Equations

Version 1.1

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Contents

Preface

Version 1

This document started of as notes for my own work. After getting too frustrated by not being able to find auxiliary formulas in my (many) notebooks and having to derive them repeatedly, I decided to compile a neat reference for myself. This is very much a work in progress. This version contains some gaping holes and is admittedly sloppy in many respects. In particular, watch out for these issues:

- The Taylor series might not hold for all x in the radius of convergence. For example, the expansion of the Gamma function only converges to the Gamma function for integer x. In addition, the radius of convergence has been omitted everywhere.
- In many cases the constant difference has been taken as $h = 1$. This has been indicated in some places, but not in all.
- In the section on standard functions of difference operators, I have not specified all the conditions for theorems where inverse functions are involved (these work "straight out of the box" only for bijections).
- The z-transform table contains some formulas that have been formally derived, without checking whether the series (of the definition) actually converges. This has been indicated by little question marks next to the transformation arrows. The range of convergence has been omitted everywhere.

Furthermore, I used some non-standard notations, and there are many inconsistencies in style or form that make this reference hard to read.

I intend to address these in coming versions; in the mean time, even with all its flaws, some might still find this reference useful.

Version 1.1

- Added Exponential Sums to differences and sums.
- Additions to the z-transform table.
- Added Binomial Transform pairs.

1. Introduction

1.1. Introduction

This document contains tables and formulas useful for working with functional equations, especially difference equations, and to a lessor extent, quotient equations.

1.2. Constants

Constants can generally be replaced by any function with appropriate properties. For instance, the table gives

$$\sum a^x = \frac{a^x}{a-1} + C.$$

Since, for any period function $C(x)$ with period 1 we have $\Delta C(x) = C(x+1) - C(x) = C(x) - C(x) = 0$, we can actually replace the constant C with any function with period 1, that is

$$\sum a^x = \frac{a^x}{a-1} + C(x).$$

Similarly, the table gives:

$$\mathrm{P}^{-1}\, x = \frac{x}{h-1} + C.$$

Here we can replace C with any function that satisfies $C(hx) = C(x)$. If $\Theta(x)$ is a periodic function with period 1, then $\Theta(\log_h x)$, satisfies this condition, so:

$$\mathrm{P}^{-1}\, x = \frac{x}{h-1} + \Theta(\log_h x).$$

In the general case, the constant for operator Δ_g can be replaced by $C(I_\Delta g(x))$, where C is any periodic function with period 1 (see formula 7.9).

2. Functions and Sequences

2.1. Discrete Power Function

$$x^{\langle a \rangle} = \frac{\Gamma(x+1)}{\Gamma(x+1-a)} \tag{2.1}$$

For integer n:

$$x^{\langle n \rangle} = x(x-1)(x-2)...(x-n+1) \tag{2.2}$$

$$x^{\langle -n \rangle} = \frac{1}{(x+1)(x+2)...(x+n)} \tag{2.3}$$

$$*x^{\langle a \rangle}(x-a)^{\langle b \rangle} = x^{\langle a+b \rangle} \tag{2.4}$$

$$x^{\langle -a \rangle} = \frac{1}{(x+1+a)^{\langle a \rangle}} \tag{2.5}$$

$$x^{\langle a \rangle_h} = h^n \frac{\Gamma\left(\frac{x}{h}+1\right)}{\Gamma\left(\frac{x}{h}+1-a\right)} \tag{2.6}$$

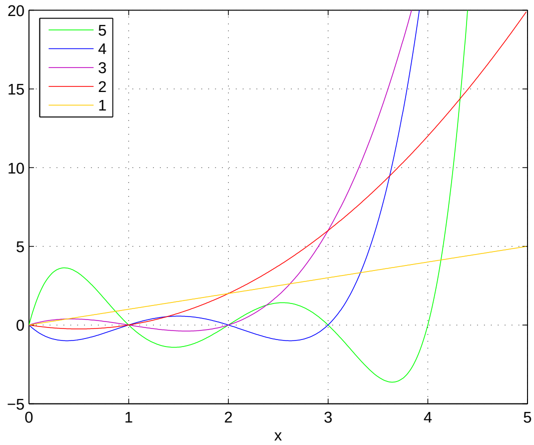
For integer n:

$$x^{\langle n \rangle_h} = x(x-h)(x-2h)...(x-(n-1)h) \tag{2.7}$$

$$x^{\langle -n \rangle_h} = \frac{1}{(x+h)(x+h)...(x+nh)} \tag{2.8}$$

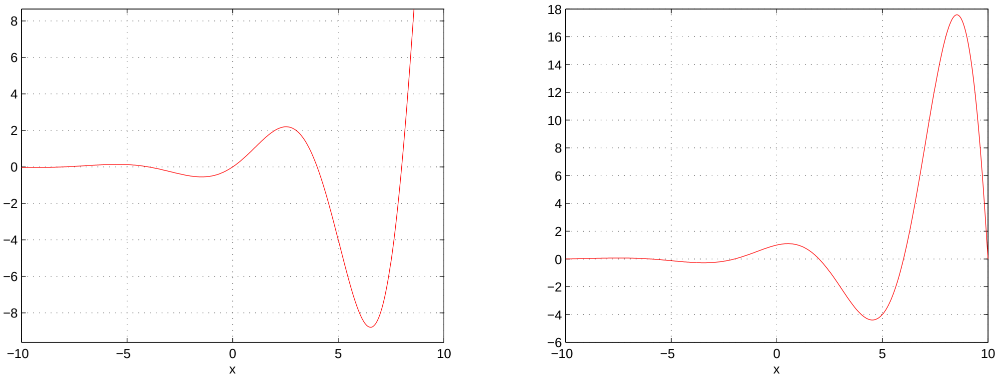
$$*x^{\langle a \rangle_h}(x-ah)^{\langle b \rangle} = x^{\langle a+b \rangle} \tag{2.9}$$

$$x^{\langle -a \rangle_h} = \frac{1}{(x+(1+a)h)^{\langle a \rangle_h}} \tag{2.10}$$



2.2. Discrete Trigonometric Functions

$$\operatorname{sind} x = 2^{x/2} \sin\left(\frac{\pi x}{4}\right) \tag{2.11}$$
$$\operatorname{cosd} x = 2^{x/2} \cos\left(\frac{\pi x}{4}\right) \tag{2.12}$$
$$\operatorname{cisd} x = 2^{x/2} \operatorname{cis}\left(\frac{\pi x}{4}\right) \tag{2.13}$$



(a) $\operatorname{sind} x$ (b) $\operatorname{cosd} x$

Figure 2.1.: Discrete Trigonometric Functions

2.2.1. Properties of Discrete Trigonometric Functions

$$\operatorname{sind}(x+y) = \operatorname{sind}(x)\operatorname{cosd}(y) + \operatorname{cosd}(x)\operatorname{sind}(y) \tag{2.14}$$
$$\operatorname{sind}(x-y) = 2^y\operatorname{sind}(x)\operatorname{cosd}(y) - 2^y\operatorname{cosd}(x)\operatorname{sind}(y) \tag{2.15}$$
$$\operatorname{cosd}(x+y) = \operatorname{cosd}(x)\operatorname{cosd}(y) - \operatorname{sind}(x)\operatorname{sind}(y) \tag{2.16}$$
$$\operatorname{cosd}(x-y) = 2^y\operatorname{cosd}(x)\operatorname{cosd}(y) + 2^y\operatorname{sind}(x)\operatorname{sind}(y) \tag{2.17}$$

$$\operatorname{sind}^2(x) + \operatorname{cosd}^2(x) = 2^x \tag{2.18}$$

$$\operatorname{sind}(2x) = 2\operatorname{sind}(x)\operatorname{cosd}(x) \tag{2.19}$$
$$\operatorname{cosd}(2x) = \operatorname{cosd}^2(x) - \operatorname{sind}^2(x) \tag{2.20}$$
$$= 2\operatorname{cosd}^2(x) - 2^x \tag{2.21}$$
$$= 2^x - 2\operatorname{sind}^2(x) \tag{2.22}$$

2.3. Fibonacci Function

$$\mathcal{F}(x) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^x - \left(\frac{1-\sqrt{5}}{2}\right)^x\right] \tag{2.23}$$

2.3.1. Properties of the Fibonacci Function

$$\mathcal{F}(x) = \mathcal{F}(x-1) + \mathcal{F}(x-2) \tag{2.24}$$

$$\mathcal{F}(x) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{F}(x-k-m) \tag{2.25}$$

2.4. G

G is the sum of the Gamma function.

$$G(x+1) = \sum_{k=0}^{\infty} !k \frac{x^{\langle k \rangle}}{k!} \qquad !n = 1, 0, 1, 2, 9, 44, 265, 1854 \tag{2.26}$$

2.5. The Hyper Power Function

A function that satisfies

$$H_a(x+1) = a^{H_a(x)} \tag{2.27}$$

and

$$H_a(0) = 1. \tag{2.28}$$

An inverse hyperpower function satisfies

$$\operatorname{slog}_a(a^x) = \operatorname{slog}_a(x) + 1. \tag{2.29}$$

Part I.

Discrete Calculus Tables

3. Sum $(x+h)$

3.1. Definition and Properties

3.1.1. Definition

$$\Delta f(x) = f(x+h) - f(x) \tag{3.1}$$

$$\sum = \Delta^{-1} \tag{3.2}$$

3.2. Properties of Operators

$$\Delta(af(x) + bg(x)) = a\Delta f(x) + b\Delta g(x) \quad \text{Linearity} \tag{3.3}$$

$$\Delta(f(x)g(x)) = f(x+1)\Delta g(x) + g(x)\Delta f(x) \quad \text{Product Rule} \tag{3.4}$$

$$\Delta \sum f(x) = f(x) \quad \text{Linearity} \tag{3.5}$$

$$\sum_{a}^{b-1} f(x) = \sum f(x)\Big|_a^b \quad \text{Fundamental Theorem} \tag{3.6}$$

$$\sum f(x)g(x) = f(x)\sum g(x) - \sum \Delta f(x) \sum g(x+1) \quad \text{Summation by Parts} \tag{3.7}$$

3.2.1. Analogues of Elementary Functions

Elementary Function	Analog
x^a	$x^{\langle a \rangle}$
e^x	2^x
$\sin x$	$\operatorname{sind} x$
$\cos x$	$\operatorname{cosd} x$

3.3. Differences

3.3.1. Powers

$$\Delta c = 0 \tag{3.8}$$

$$\Delta ax = ah \tag{3.9}$$

$$\Delta x^2 = 2hx + h^2 \tag{3.10}$$

$$\Delta x^3 = 3hx^2 + 3h^2x + h^3 \tag{3.11}$$

$$\Delta x^n = \sum_{k=0}^{n-1} \binom{n}{k} x^k h^{n-k} \tag{3.12}$$

$$\Delta x^{\langle a \rangle_h} = ax^{\langle a-1 \rangle_h} \tag{3.13}$$

3.3.2. Binomial Coefficients

$$\Delta\binom{x}{n}_h = h\binom{x}{n-1}_h \tag{3.14}$$

$$\Delta\binom{-x}{n} = \binom{-x-1}{n-1} \qquad h=1 \tag{3.15}$$

3.3.3. Exponential Functions

$$\Delta a^x = a^x(a^h-1) \tag{3.16}$$

$$\Delta \log_a x = \log_a\left(1+\frac{h}{x}\right) \tag{3.17}$$

$$\Delta a^{\langle x\rangle} = a^{\langle x\rangle}(x+1-a) \qquad h=1 \tag{3.18}$$

$$\Delta\mathcal{F}(x) = \mathcal{F}(x-1) \qquad h=1 \tag{3.19}$$

3.3.4. Trigonometric Functions

$$\Delta\sin ax = 2\sin\frac{ah}{2}\cos\left(ax+\frac{ah}{2}\right) \tag{3.20}$$

$$\Delta\cos ax = -2\sin\frac{ah}{2}\sin\left(ax+\frac{ah}{2}\right) \tag{3.21}$$

$$\Delta\tan ax = \frac{\sin ah}{\cos(ax+ah)\cos ax} \tag{3.22}$$

$$\Delta\cot ax = \frac{\sin ah}{\sin(ax+ah)\sin ax} \tag{3.23}$$

$$\Delta\sec x = \frac{\sin(ax+\frac{ah}{2})\sin(\frac{ah}{2})}{\cos(ax+ah)\cos ax} \tag{3.24}$$

$$\Delta\csc x = -\frac{\cos(ax+\frac{ah}{2})\sin(\frac{ah}{2})}{\sin(ax+ah)\sin ax} \tag{3.25}$$

3.3.5. Inverse Trigonometric Functions

$$\Delta\operatorname{asin} ax = \operatorname{asin}\left(a(x+h)\sqrt{1-a^2x^2} - ax\sqrt{1-a^2(x+h)^2}\right) \tag{3.26}$$

$$\Delta\operatorname{acos} ax = \operatorname{acos}\left(a^2x(x+h) - \sqrt{(1-a^2(x+h)^2)(1-a^2x^2)}\right) \tag{3.27}$$

$$\Delta\operatorname{atan} ax = \operatorname{atan}\left(\frac{ah}{a^2x^2+a^2xh+1}\right) \tag{3.28}$$

$$\Delta\operatorname{acot} ax = -\operatorname{acot}\left(\frac{a^2x^2+a^2xh+1}{ah}\right) \tag{3.29}$$

$$\Delta\operatorname{asec} ax = -\operatorname{asec}\left(\frac{a^2x(x+h)}{1-\sqrt{(a^2x^2-1)(a^2(x+h)^2-1)}}\right) \tag{3.30}$$

$$\Delta\operatorname{acsc} ax = \operatorname{asec}\left(\frac{a^2x(x+h)\sqrt{(1-a^2x^2)(1-a^2(x+h)^2)}}{ax\sqrt{1-a^2(x+h)^2}+a(x+h)\sqrt{1-a^2x^2}}\right) \tag{3.31}$$

3.3.6. Hyperbolic Functions

$$\Delta \sinh x = 2\cosh\left(x+\frac{h}{2}\right)\sinh\left(\frac{h}{2}\right) \tag{3.32}$$

$$\Delta \cosh x = 2\sinh\left(x+\frac{h}{2}\right)\sinh\left(\frac{h}{2}\right) \tag{3.33}$$

$$\Delta \tanh x = \frac{\sinh h}{\cosh(x+h)\cosh(x)} \tag{3.34}$$

$$\Delta \coth x = -\frac{\sinh h}{\sinh(x+h)\sinh(x)} \tag{3.35}$$

$$\Delta \operatorname{sech} x = -\frac{2\sinh(x+\frac{h}{2})\sinh(\frac{h}{2})}{\cosh(x+h)\cosh(x)} \tag{3.36}$$

$$\Delta \operatorname{csch} x = -\frac{2\cosh(x+\frac{h}{2})\sinh(\frac{h}{2})}{\sinh(x+h)\sinh(x)} \tag{3.37}$$

3.3.7. Exponential Sums

$$\Delta \operatorname{Bs}(x) = \frac{2^x}{x} \tag{3.38}$$

$$\Delta \operatorname{Ss}(x) = \frac{\operatorname{sind} x}{x} \tag{3.39}$$

$$\Delta \operatorname{Cs}(x) = \frac{\operatorname{cosd} x}{x} \tag{3.40}$$

$$\Delta \operatorname{Shs}(x) = \frac{\operatorname{sinhd} x}{x} \tag{3.41}$$

$$\Delta \operatorname{Chs}(x) = \frac{\operatorname{coshd} x}{x} \tag{3.42}$$

3.3.8. Discrete Additive Trigonometric Functions

$$\Delta \operatorname{sind} x = \operatorname{cosd} x \tag{3.43}$$

$$\Delta \operatorname{cosd} x = -\operatorname{sind} x \tag{3.44}$$

3.3.9. Gamma and Related Functions

Here, $h = 1$. The labels (HMF x) refers to the number in the *Handbook of Mathematical Functions* where special functions have been defined. The function $G(x)$ is defined by formula 2.26.

$$\Delta\Gamma(x) = (x-1)\Gamma(x) \tag{3.45}$$

$$\Delta G(x) = \Gamma(x) \tag{3.46}$$

$$\Delta xG(x) = G(x+2) \tag{3.47}$$

$$\Delta B(x,a) = \frac{-a}{x+a}B(x,a) \tag{3.48}$$

$$\Delta\varphi_0(x) = \frac{1}{x} \tag{3.49}$$

$$\Delta\varphi_0'(x) = -\frac{1}{x^2} \tag{3.50}$$

$$\Delta\varphi_n(x) = \frac{(-1)^n n!}{x^{n+1}} \tag{3.51}$$

$$\Delta P(x,a) = \frac{-a^x e^{-a}}{\Gamma(x+1)} \qquad \text{(HMF 6.5.1)} \tag{3.52}$$

$$\Delta\gamma(x,a) = (x-1)\gamma(x,a) - a^x e^a \qquad \text{(HMF 6.5.2)} \tag{3.53}$$

$$\Delta\Gamma(x,a) = (x-1)\Gamma(x,a) + a^x e^a \qquad \text{(HMF 6.5.3)} \tag{3.54}$$

$$\Delta\gamma^*(x,a) = \gamma^*(x,a) - \frac{1}{e^a\Gamma(x+1)} \qquad \text{(HMF 6.5.4)} \tag{3.55}$$

$$\Delta I_t(x,a) = (t-1)I_t(x,a) + (t-1)I_t(x,a-1) \qquad \text{(HMF 6.6.2)} \tag{3.56}$$

3.3.10. More Exponential Forms

$$\Delta 2^x \frac{a^{2^x}+1}{a^{2^x}-1} = 2^x \frac{a^{2^x}-1}{a^{2^x}+1} \tag{3.57}$$

3.3.11. Forms involving $\sin x$ and 2^x.

$$\Delta 2^x \sin\left(\frac{a}{2^x}\right) = \sin\left(\frac{a}{2^{x+1}}\right)\sin^2\left(\frac{a}{2^{x+2}}\right) \tag{3.58}$$

$$\Delta 2^x \sin^2\left(\frac{a}{2^x}\right) = 2^{2x+2}\sin^4\left(\frac{a}{2^{x+1}}\right) \tag{3.59}$$

$$\Delta \tan\left(\frac{a}{2^x}\right) = -\frac{\tan\left(\frac{a}{2^{x+1}}\right)}{\cos\left(\frac{a}{2^x}\right)} \tag{3.60}$$

$$\Delta(-2)^x \sin\left(\frac{a}{2^x}\right) = (-2)^{x+3}\sin\left(\frac{a}{2^{x+1}}\right)\cos\left(\frac{a}{2^{x+1}}\right) \tag{3.61}$$

$$\Delta\frac{1}{2^x\sin\left(\frac{a}{2^x}\right)} = -2\frac{\sin^2\left(\frac{a}{2^{x+2}}\right)}{2^x\sin\left(\frac{a}{2^x}\right)} \tag{3.62}$$

$$\Delta\frac{1}{2^x\tan\left(\frac{a}{2^x}\right)} = \frac{\tan\left(\frac{a}{2^{x+1}}\right)}{2^{x+1}} \tag{3.63}$$

$$\Delta\left(\frac{1}{2^x\tan\left(\frac{a}{2^x}\right)}\right)^2 = \frac{2}{2^x} - \left(\frac{\tan\left(\frac{a}{2^{x+1}}\right)}{2^{x+1}}\right) \tag{3.64}$$

$$\Delta\frac{\cos\left(\frac{a}{2^x}\right)}{2^x\sin\left(\frac{a}{2^x}\right)} = \frac{\sin\left(\frac{a}{2^x}\right)}{\left(2^{x+1}\cos\frac{a}{2^{x+1}}\right)} \tag{3.65}$$

$$\Delta\cot(2^x a) = -\frac{1}{\sin\left(2^{x+1}a\right)} \tag{3.66}$$

$$\Delta\frac{\ln(2\sin 2^x a)}{2^x} = -\frac{\log\tan 2^x a}{2^{x+1}} \tag{3.67}$$

$$\Delta\frac{1}{\left(2^x\sin\left(\frac{a}{2^x}\right)\right)^2} = -\frac{1}{\left(2^{x+1}\cos\left(\frac{a}{2^{x+1}}\right)\right)^2} \tag{3.68}$$

$$\Delta\frac{1}{\prod_{k=0}^{2n+1}\cos(a(x+k))} = 2\sin(a(n+1))\frac{\sin(a(x+n+1))}{\prod_{k=0}^{2n+2}\cos(a(x+k))} \tag{3.69}$$

$$\Delta\frac{(-1)^x}{\sin(ax)\sin(a(x+1))} = 2\cos a\frac{(-1)^{x+1}}{\sin(ax)\sin(a(x+2))} \tag{3.70}$$

$$\Delta\frac{(-1)^x}{\prod_{k=0}^{2n+1}\sin(a(x+k))} = 2\cos(a(n+1))\frac{(-1)^{x+1}\sin(a(x+n+1)}{\prod_{k=0}^{2n+2}\sin(a(x+k))} \tag{3.71}$$

$$\Delta\frac{(-1)^x}{\cos(ax)\cos(a(x+1))} = 2\cos a\frac{(-1)^{x+1}}{\cos(ax)\cos(a(x+2))} \tag{3.72}$$

$$\Delta\frac{(-1)^x}{\prod_{k=0}^{2n+1}\cos(a(x+k))} = 2\cos(a(n+1))\frac{(-1)^{x+1}\cos(a(x+2n+1))}{\prod_{k=0}^{2n+2}\cos(a(x+k))} \tag{3.73}$$

3.3.12. Forms involving $\operatorname{atan} x$

$$\Delta\operatorname{atan}\left(\frac{ax+b}{cx+d}\right) = \operatorname{atan}\left(\frac{bc-ad}{a^2+ab+c^2+cd+(2ab+b^2+2cd+d^2)x+(b^2+d^2)x^2}\right) \tag{3.74}$$

$$\Delta\operatorname{atan}(xf(x)) = \operatorname{atan}\left(\frac{\Delta f(x)}{1+f(x)f(x+1)}\right) \tag{3.75}$$

$$\Delta 2^x\operatorname{atan}\left(\frac{a}{2^x}\right) = 2^x\operatorname{atan}\left(\frac{a^3}{2^{3x+2}+3a^2 2^x}\right) \tag{3.76}$$

3.4. Sums

3.4.1. Basics

$$\sum a = ahx + C \tag{3.77}$$

$$\sum (-1)^x \binom{n}{x} = (-1)^{x+1}\binom{n-1}{x-1} + C \tag{3.78}$$

$$\sum x = \frac{1}{2}x^2 - \frac{h}{2}x + C \tag{3.79}$$

$$\sum x^2 = \frac{1}{3}x^3 - \frac{2h}{3}x^2 - \frac{h^2}{6}x + C \tag{3.80}$$

$$\sum x^3 = \frac{1}{4}x^4 - \frac{h}{2}x^3 + \frac{h^2}{4}x^2 + C \tag{3.81}$$

$$\sum x^4 = \frac{1}{5}x^5 - \frac{h}{2}x^4 + \frac{h^2}{3}x^3 - \frac{h^4}{30}x + C \tag{3.82}$$

$$\sum a^x = \frac{h}{a^h - 1}a^x + C \tag{3.83}$$

$$\sum \log_a x = \log_a \Gamma(x) + C \qquad h = 1 \tag{3.84}$$

$$?\sum x^n = \sum_{k=1}^{n} \frac{S_1(n,k)}{k+1} x^{\langle k+1 \rangle} + C \qquad h = 1 \tag{3.85}$$

$$\sum x^n = \frac{B_{n+1}(x)}{n+1} + C \qquad h = 1 \tag{3.86}$$

$$\sum x^n = \frac{E_n(x)}{2} + C \qquad h = 1 \tag{3.87}$$

$$\sum x^{\langle n \rangle} = \frac{x^{\langle n \rangle}}{n+1} + C \qquad h = 1 \tag{3.88}$$

$$\sum \frac{1}{x} = \varphi_0(x) + C \qquad h = 1 \tag{3.89}$$

$$\sum \frac{1}{x^n} = \frac{(-1)^{n-1}\varphi_{n-1}(x)}{(n-1)!} + C \qquad h = 1 \tag{3.90}$$

3.4.2. Trigonometric Functions

$$\sum \sin(2ax) = -\frac{\cos[a(2x-1)]}{2\sin a} \qquad a \neq \pi n \tag{3.91}$$

$$\sum \sin(2\pi nx) = x\sin(2\pi nx) \tag{3.92}$$

$$\sum \cos(2ax) = \frac{\sin[a(2x-1)]}{2\sin a} \qquad a \neq \pi n \tag{3.93}$$

$$\sum \cos(2\pi nx) = x\cos(2\pi nx) \tag{3.94}$$

$$\sum \sin^2(ax) = \frac{x}{2} - \frac{\sin[a(2x-1)]}{4\sin a} \qquad a \neq \pi n \tag{3.95}$$

$$\sum \sin^2(\pi nx) = x\sin^2(2\pi nx) \tag{3.96}$$

$$\sum \cos^2(ax) = \frac{x}{2} + \frac{\sin[a(2x-1)]}{4\sin a} \qquad a \neq \pi n \tag{3.97}$$

$$\sum \cos^2(\pi nx) = x\cos^2(2\pi nx) \tag{3.98}$$

$$\sum x\sin(2ax) = \frac{\sin[a(2x-1)]}{4\sin^2 a} - x\frac{\cos[a(2x-1)]}{2\sin a} \qquad a \neq \pi n \tag{3.99}$$

$$\sum x\sin(2\pi nx) = \frac{1}{2}x(x-1)\sin^2(2\pi nx) \tag{3.100}$$

$$\sum x\cos(2ax) = \frac{\cos[a(2x-1)]}{4\sin^2 a} + x\frac{\sin[a(2x-1)]}{2\sin a} \qquad a \neq \pi n \tag{3.101}$$

$$\sum x\cos(2\pi nx) = \frac{1}{2}x(x-1)\cos^2(2\pi nx) \tag{3.102}$$

$$\sum (-1)^x\cos(2bx) = (-1)^{x+1}\frac{\cos[b(2x-1)]}{2\cos b} \tag{3.103}$$

$$\sum (-1)^x\sin(2bx) = (-1)^{x+1}\frac{\sin[b(2x-1)]}{2\cos b} \tag{3.104}$$

$$\sum a^x\sin(bx) = a^x\frac{a\sin[b(x-1)] - \sin(bx)}{a^2 - 2a\cos b + 1} \qquad a > 0, \quad a \neq 1 \tag{3.105}$$

$$\sum a^x\cos(bx) = a^x\frac{a\cos[b(x-1)] - \cos(bx)}{a^2 - 2a\cos b + 1} \qquad a > 0, \quad a \neq 1 \tag{3.106}$$

$$\sum \tan\pi x = (x-1)\tan\pi x \tag{3.107}$$

$$\sum \cot\pi x = (x-1)\cot\pi x \tag{3.108}$$

3.4.3. Hyperbolic Functions

$$\sum \sinh(2ax) = \frac{\cosh\left[a(2x-1)\right]}{2\sinh(a)} \qquad a \neq \pi n \qquad (3.109)$$

$$\sum \sinh(2\pi nx) = x\sinh(2\pi nx) \qquad (3.110)$$

$$\sum \cosh(2ax) = \frac{\sinh\left[a(2x-1)\right]}{2\sinh(a)} \qquad a \neq \pi n \qquad (3.111)$$

$$\sum \cosh(2\pi nx) = x\cosh(2\pi nx) \qquad (3.112)$$

$$\sum \sinh^2(ax) = \frac{\sinh\left[a(2x-1)\right]}{4\sinh(a)} - \frac{x}{2} \qquad a \neq \pi n \qquad (3.113)$$

$$\sum \sinh^2(\pi nx) = x\sinh^2(2\pi nx) \qquad (3.114)$$

$$\sum \cosh^2(ax) = \frac{\sinh\left[a(2x-1)\right]}{4\sinh(a)} + \frac{x}{2} \qquad a \neq \pi n \qquad (3.115)$$

$$\sum \cosh^2(\pi nx) = x\cosh^2(2\pi nx) \qquad (3.116)$$

$$\sum x\sinh(2ax) = \frac{\sinh[a(2x-1)]}{4\sinh^2 a} - x\frac{\cosh[a(2x-1)]}{2\sinh a} \qquad a \neq \pi n \qquad (3.117)$$

$$\sum x\sinh(2\pi nx) = \frac{1}{2}x(x-1)\sinh^2(2\pi nx) \qquad (3.118)$$

$$\sum x\cosh(2ax) = \frac{\cosh[a(2x-1)]}{4\sinh^2 a} + x\frac{\sinh[a(2x-1)]}{2\sinh a} \qquad a \neq \pi n \qquad (3.119)$$

$$\sum x\cosh(2\pi nx) = \frac{1}{2}x(x-1)\cosh^2(2\pi nx) \qquad (3.120)$$

$$\sum (-1)^x\cosh(2bx) = (-1)^x\frac{\cosh[b(2x-1)]}{2\cosh b} \qquad (3.121)$$

$$?\sum (-1)^x\sinh(2bx) = (-1)^x\frac{\sinh[b(2x-1)]}{2\cosh b} \qquad (3.122)$$

$$\sum a^x\sinh(bx) = a^x\frac{a\sinh[b(x-1)] - \sinh(bx)}{a^2 - 2a\cosh b + 1} \qquad a > 0, \quad a \neq 1 \qquad (3.123)$$

$$\sum a^x\cosh(bx) = a^x\frac{a\cosh[b(x-1)] - \cosh(bx)}{a^2 - 2a\cosh b + 1} \qquad a > 0, \quad a \neq 1 \qquad (3.124)$$

$$\sum \tanh \pi ix = (x-1)\tanh \pi ix \qquad (3.125)$$

$$\sum \coth \pi ix = (x-1)\coth \pi ix \qquad (3.126)$$

3.4.4. Exponential Sums

$$\sum \frac{2^x}{x} = \mathrm{Bs}(x) \qquad (3.127)$$

$$\sum \frac{\operatorname{sind} x}{x} = \mathrm{Ss}(x) \qquad (3.128)$$

$$\sum \frac{\operatorname{cosd} x}{x} = \mathrm{Cs}(x) \qquad (3.129)$$

$$\sum \frac{\operatorname{sinhd} x}{x} = \mathrm{Shs}(x) \qquad (3.130)$$

$$\sum \frac{\operatorname{coshd} x}{x} = \mathrm{Chs}(x) \qquad (3.131)$$

3.4.5. Gamma Functions

$$\sum \Gamma(x) = G(x) + C \tag{3.132}$$
$$\sum G(x) = (x-2)G(x-2) + C \tag{3.133}$$
$$\sum \varphi_0(x) = (x-1)\varphi_0(x) - x + C \tag{3.134}$$
$$\sum \varphi_1(x) = (x-1)\varphi_1(x) + \varphi_0(x) + C \tag{3.135}$$
$$\sum \varphi_n(x) = (x-1)\varphi_n(x) + (-1)^{n-1}\varphi_{n-1}(x) + C \tag{3.136}$$

3.5. Taylor Series

$$x^n = \sum_{k=0}^{n} S_1(n,k)x^{\langle k\rangle} \tag{3.137}$$
$$a^x = \sum_{k=0}^{\infty} (a-1)^k \frac{x^{\langle k\rangle}}{k!} \tag{3.138}$$
$$\log_a(x+1) = \sum_{k=0}^{\infty}\sum_{m=0}^{k} (-1)^{m+k}\binom{k}{m}\log_a(m+1)\frac{x^{\langle k\rangle}}{k!} \tag{3.139}$$

Here we define:

$$\operatorname{soc}_k(x) = \begin{cases} \sin(x) & k \text{ is even} \\ \cos(x) & k \text{ is odd} \end{cases}$$

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^{\lfloor \frac{k}{2} \rfloor} \sin^k\left(\frac{1}{2}\right) \operatorname{soc}_k\left(\frac{n}{2}\right)\frac{x^{\langle k\rangle}}{k!} \tag{3.140}$$
$$\cos(x) = \sum_{k=0}^{\infty} (-1)^{\lfloor \frac{k+1}{2} \rfloor} \sin^k\left(\frac{1}{2}\right) \operatorname{soc}_{k+1}\left(\frac{n}{2}\right)\frac{x^{\langle k\rangle}}{k!} \tag{3.141}$$
$$\operatorname{sind}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{\langle 2k+1\rangle}}{(2k+1)!} \tag{3.142}$$
$$\operatorname{cosd}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{\langle 2k\rangle}}{(2k)!} \tag{3.143}$$

$$\Gamma(x+1) = \sum_{k=0}^{\infty} !(k+1)\frac{x^{\langle k\rangle}}{k!} \qquad !n = 1, 0, 1, 2, 9, 44, 265, 1854 \tag{3.144}$$
$$G(x+1) = \sum_{k=0}^{\infty} !k\frac{x^{\langle k\rangle}}{k!} \qquad !n = 1, 0, 1, 2, 9, 44, 265, 1854 \tag{3.145}$$
$$\varphi_0(x+1) = \gamma + \sum_{k=1}^{\infty} (-1)^k (k-1)! x^{\langle k\rangle} \tag{3.146}$$
$$\frac{1}{x+1} = \sum_{k=0}^{\infty} (-1)^{k+1}(k+1)! x^{\langle k\rangle} \tag{3.147}$$

$$\mathrm{Bs}(x+1) = \varphi_0(x+1) + \sum_{k=1}^{\infty} \frac{1}{k} \cdot \frac{x^{\langle k \rangle}}{k!} \tag{3.148}$$

$$= \gamma + \sum_{k=1}^{\infty} \frac{(-1)^k (k!)^2 + 1}{k \cdot k!} x^{\langle k \rangle} \tag{3.149}$$

$$\mathrm{Ss}(x+1) = \varphi_0(x+1) + \sum_{k=1}^{\infty} (-1)^k \frac{1}{2k+1} \cdot \frac{x^{\langle 2k+1 \rangle}}{(2k+1)!} \tag{3.150}$$

$$= \gamma + \sum_{k=1}^{\infty} (-1)^k \frac{(2k+1) \cdot (2k+1)!(k-1)! + 1}{(2k+1) \cdot (2k+1)!} x^{\langle k \rangle} \tag{3.151}$$

$$\mathrm{Cs}(x+1) = \varphi_0(x+1) + \sum_{k=1}^{\infty} (-1)^k \frac{1}{2k} \cdot \frac{x^{\langle 2k \rangle}}{(2k)!} \tag{3.152}$$

$$= \gamma + \sum_{k=1}^{\infty} (-1)^k \frac{2k \cdot (2k)!(k-1)! + 1}{2k \cdot (2k)!} x^{\langle k \rangle} \tag{3.153}$$

$$\tag{3.154}$$

3.6. Miscellaneous Difference Identities

$$\Delta \mathcal{F}(x) = \mathcal{F}(x-1) \tag{3.155}$$

$$\Delta W(x) = \Delta \ln x - \Delta \ln W(x) \tag{3.156}$$

3.7. Difference Equations

3.7.1. $p(x)f(x+1) - f(x) = q(x)$

$$f(x) = \frac{\sum (q(x) \prod p(x))}{\prod p(x)}$$

3.7.2. $f(x)\Delta f(x) = 1$

$$f(x) = \frac{c\mathcal{F}(x) - \mathcal{F}(x-1)}{c\mathcal{F}(x-1) - \mathcal{F}(x-2)}$$

4. Sum (xh)

4.1. Analogues of Elementary Functions

Elementary Function	Analog
x^a	$(\log_h x)^{\langle a \rangle}$
e^x	-
$\sin x$	-
$\cos x$	-

4.2. Properties of Operators

$$\mathrm{P}(af(x) + bg(x)) = a\,\mathrm{P}\,f(x) + b\,\mathrm{P}\,g(x) \tag{4.1}$$

$$\mathrm{P}(f(x)g(x)) = f(hx)\,\mathrm{P}\,g(x) + g(x)\,\mathrm{P}\,f(x) \tag{4.2}$$

$$\mathrm{P}\,\mathrm{P}^{-1}\,f(x) = f(x) \tag{4.3}$$

$$\sum_{a}^{b-1} f(h^{x-a}x) = \mathrm{P}^{-1}\,f(x)\big|_a^b \tag{4.4}$$

$$\mathrm{P}^{-1}\,f(x)g(x) = f(x)\,\mathrm{P}^{-1}\,g(x) - \mathrm{P}^{-1}\,\mathrm{P}\,f(x)\,\mathrm{P}^{-1}\,g(x+1) \tag{4.5}$$

$$\mathrm{P}^{-1}\,xf(x) = xD\,\mathrm{P}^{-1}\int f(x) \tag{4.6}$$

4.3. Differences

4.3.1. Basics

$$\mathrm{P}\,c = 0 \tag{4.7}$$

$$\mathrm{P}\,x^a = (h^a - 1)x^a \tag{4.8}$$

$$\mathrm{P}\log_a x = \log_a h \tag{4.9}$$

$$\mathrm{P}(\log_h x)^{\langle n \rangle} = n(\log_h x)^{\langle n-1 \rangle} \tag{4.10}$$

$$\mathrm{P}\,a^x = a^x(a^{(h-1)x} - 1) \tag{4.11}$$

$$\mathrm{P}\,2^{\log_h x} = 2^{\log_h x} \tag{4.12}$$

4.4. Sums

4.4.1. Basics

$$\mathrm{P}^{-1}\, a = a \log_h x \tag{4.13}$$

$$\mathrm{P}^{-1}\, x^a = \frac{x^a}{h^a - 1} \tag{4.14}$$

$$\mathrm{P}^{-1} \log_a x = \frac{\log_a x(\log_h x - 1)}{2} \tag{4.15}$$

$$\mathrm{P}^{-1} (\log_h x)^{\langle n\rangle} = \frac{(\log_h x)^{\langle n+1\rangle}}{n+1} \tag{4.16}$$

$$\mathrm{P}^{-1} (\log_a x)^{\langle n\rangle} = \frac{\log_a x(\log_h x - 1)^{\langle n\rangle}}{n+1} \tag{4.17}$$

4.4.2. Trigonometric Functions

$$\mathrm{P}^{-1} \sin(x) = S(x,h) = \sum_{k=1}^{\infty} \sin(h^{-k}x) \tag{4.18}$$

$$\mathrm{P}^{-1} \cos(x) = C(x,h) = \log_h x + \sum_{k=1}^{\infty} (\cos(h^{-k}x) - 1) \tag{4.19}$$

$$\mathrm{P}^{-1}\, x\cos(x) = xS'(x,h) \tag{4.20}$$

$$\mathrm{P}^{-1}\, x\sin(x) = -xC'(x,h) \tag{4.21}$$

$$\tag{4.22}$$

5. Product $(x+1)$

5.1. Analogues of Elementary Functions

Elementary Function	Analog
x^n	$\exp\left(x^{\langle n\rangle}\right)$
x^{-n}	$\exp\left(x^{\langle -n\rangle}\right)$
e^x	$\exp(2^x)$
$\sin x$	$\exp(\operatorname{sind} x)$
$\cos x$	$\exp(\operatorname{cosd} x)$

5.2. Properties of Operators

$$\mathrm{Q}(f(x)^a g(x)^b) = (\mathrm{Q}\, f(x))^a (\mathrm{Q}\, g(x))^b \tag{5.1}$$

$$\mathrm{Q}\, f(x)^{g(x)} = \left[\mathrm{Q}\, e^{g(x)}\right]^{\ln f(x+1)} [\mathrm{Q}\, f(x)]^{g(x)} \tag{5.2}$$

$$\mathrm{Q}\, f(x)^{g(x)} = \left[\mathrm{Q}\, e^{g(x)}\right]^{\ln f(x)} [\mathrm{Q}\, f(x)]^{g(x+1)} \tag{5.3}$$

$$\mathrm{Q} \prod f(x) = f(x) \tag{5.4}$$

$$\prod_a^{b-1} f(x) = \frac{F(b)}{F(a)}, \quad F(x) = \prod f(x) \tag{5.5}$$

$$\prod f(x)^{g(x)} = \frac{\left(\prod f(x)\right)^{g(x)}}{\prod\left(\prod f(x+1)\right)^{\Delta g(x)}} \tag{5.6}$$

5.2.1. Relation with Additive Operators

$$a^{\Delta f(x)} = \mathrm{Q}\, a^{f(x)} \tag{5.7}$$

$$a^{\sum f(x)} = \prod a^f(x) \tag{5.8}$$

5.3. Quotients

5.3.1. Basics

$$\mathrm{Q}\, c = 1 \tag{5.9}$$
$$\mathrm{Q}\, x = 1 + \frac{1}{x} \tag{5.10}$$
$$\mathrm{Q}\, x^2 = 1 + \frac{2}{x} + \frac{1}{x^2} \tag{5.11}$$
$$\mathrm{Q}\, x^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{x^k} \tag{5.12}$$
$$\mathrm{Q}\, x^{\langle a \rangle} = \frac{x+1}{x-a+1} \tag{5.13}$$
$$\mathrm{Q}\, a^x = a \tag{5.14}$$
$$\mathrm{Q}\, a^{2^x} = a^{2^x} \tag{5.15}$$
$$\mathrm{Q}\, a^{b^x} = a^{b^x (b-1)} \tag{5.16}$$
$$\mathrm{Q} \log_a x = \log_x (x+1) \tag{5.17}$$
$$\mathrm{Q} \exp\left(x^{\langle n \rangle}\right) = \exp\left(n x^{\langle n-1 \rangle}\right) \tag{5.18}$$

5.3.2. Trigonometric Functions

$$\mathrm{Q} \sin x = \cos 1 + \sin 1 \cot x \tag{5.19}$$
$$\mathrm{Q} \cos x = \cos 1 - \sin 1 \tan x \tag{5.20}$$
$$\mathrm{Q} \tan x = \frac{\cos 1 + \sin 1 \cot x}{\cos 1 - \sin 1 \tan x} \tag{5.21}$$
$$\mathrm{Q} \cot x = \frac{\cos 1 - \sin 1 \tan x}{\cos 1 + \sin 1 \cot x} \tag{5.22}$$
$$\mathrm{Q} \sec x = \frac{1}{\cos 1 - \sin 1 \tan x} \tag{5.23}$$
$$\mathrm{Q} \csc x = \frac{1}{\cos 1 + \sin 1 \cot x} \tag{5.24}$$
$$\tag{5.25}$$

5.3.3. Hyperbolic Functions

$$\mathrm{Q} \sinh x = \cosh 1 + \sinh 1 \coth x \tag{5.26}$$
$$\mathrm{Q} \cosh x = \cosh 1 + \sinh 1 \tanh x \tag{5.27}$$
$$\mathrm{Q} \tanh x = \frac{\cosh 1 + \sinh 1 \coth x}{\cosh 1 + \sinh 1 \tanh x} \tag{5.28}$$
$$\mathrm{Q} \coth x = \frac{\cosh 1 + \sinh 1 \tanh x}{\cosh 1 + \sinh 1 \coth x} \tag{5.29}$$
$$\mathrm{Q}\, \mathrm{sech}\, x = \frac{1}{\cosh 1 + \sinh 1 \tanh x} \tag{5.30}$$
$$\mathrm{Q}\, \mathrm{csch}\, x = \frac{1}{\cosh 1 + \sinh 1 \coth x} \tag{5.31}$$
$$\tag{5.32}$$

5.3.4. Discrete Additive Trigonometric Functions

$$\mathrm{Q}\,\mathrm{sind}\,x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cot\left(\frac{\pi x}{4}\right) \tag{5.33}$$

$$\mathrm{Q}\,\mathrm{cosd}\,x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\tan\left(\frac{\pi x}{4}\right) \tag{5.34}$$

5.3.5. Gamma and Related Functions

$$\mathrm{Q}\,\Gamma(x) = x \tag{5.35}$$

$$\mathrm{Q}\,\Gamma^n(x) = x^n \tag{5.36}$$

$$\mathrm{Q}(a^x\Gamma^n(x)) = ax^n \tag{5.37}$$

$$\mathrm{Q}\,\Gamma(x+a) = x+a \tag{5.38}$$

$$\mathrm{Q}\,B(a,x) = \frac{1}{x+a}\left(\frac{x\Gamma(x)+\Gamma(a)}{\Gamma(x)+\Gamma(a)}\right) \tag{5.39}$$

5.4. Products

5.4.1. Basics

$$\prod(ax^n) = (a^x\Gamma^n(x)) \tag{5.40}$$

$$\prod(ax+b) = a^x\Gamma\left(x+\frac{b}{a}\right) \tag{5.41}$$

$$\prod a^x = a^{x(x-1)/2} \tag{5.42}$$

$$\prod a\frac{(x-p_1)(x-p_2)\cdots(x-p_m)}{(x-q_1)(x-q_2)\cdots(x-q_n)} = a^x\frac{\Gamma(x-p_1)\Gamma(x-p_2)\cdots\Gamma(x-p_m)}{\Gamma(x-q_1)\Gamma(x-q_2)\cdots\Gamma(x-q_n)} \tag{5.43}$$

$$\prod a^{\frac{1}{x}} = a^{\varphi_0(x)} \tag{5.44}$$

5.5. Taylor Series

$$f(x) = \prod_{n=0}^{\infty} f^{\mathrm{Q}\,n}(0)\exp\left(\frac{x^{\langle n\rangle}}{n!}\right) \tag{5.45}$$

5.6. Miscellaneous Identities

$$\mathrm{Q}\,\mathcal{F}(x) = k + \mathrm{Q}\,\mathcal{F}(x-k) \tag{5.46}$$

5.7. Quotient Equations

The solution of the equation:

$$\Delta \frac{f(x+1)^{p(x)}}{f(x)} = q(x) \tag{5.47}$$

is given by

$$f(x) = \left(\prod q(x)^{\prod p(x)}\right)^{1/\prod p(x)}$$

6. Product (hx)

6.1. Definition

$$\mathrm{Q}\, f(x) = \frac{f(xh)}{f(x)}$$

6.2. Quotients

6.2.1. Basics

$$\mathrm{Q}\, c = 1 \tag{6.1}$$
$$\mathrm{Q}\, x = h \tag{6.2}$$
$$\mathrm{Q}\, x^n = h^n \tag{6.3}$$
$$\mathrm{Q}\, a^x = a^{x(h-1)} \tag{6.4}$$
$$\mathrm{Q}\log_a x = 1 + \log_x h \tag{6.5}$$
$$\mathrm{Q}\log_x a = 1 - \frac{\ln h}{\ln h + \ln x} \tag{6.6}$$

7. Generalised Operators

7.1. Definitions

$$\Delta_g f(x) = f(g(x)) - f(x) \tag{7.1}$$

$$\mathrm{Q}_g\, f(x) = \frac{f(g(x))}{f(x)} \tag{7.2}$$

7.2. Theorems

$$\Delta_g(af_1(x) + bf_2(x)) = a\Delta_g f_1(x) + b\Delta_g f_2(x) \quad \text{Linearity} \tag{7.3}$$

$$\Delta_g(f_1(x)f_2(x)) = f_1(g(x))\Delta_g f_2(x) + f_2(x)\Delta_g f_1(x) \quad \text{Product Rule} \tag{7.4}$$

$$\Delta_g(f_1(x)f_2(x)) = f_1(x)\Delta_g f_2(x) + f_2(g(x))\Delta_g f_1(x) \quad \text{Product Rule} \tag{7.5}$$

$$\sum_{k=0}^{n-1} \Delta_g f(g^k(x)) = f(g^n(x)) - f(x) \quad \text{Fundemental Theorem} \tag{7.6}$$

$$\left(\prod_g g'(x)\right) D\Delta_g f(x) = \Delta_g \left(f'(x)\prod_g g'(x)\right) \tag{7.7}$$

7.3. Standard Functions of Difference Operators

7.4. Definitions

$$I_\Delta g(x) = f(x) \Rightarrow f(g(x)) - f(x) = 1 \tag{7.8}$$

7.5. Theorems

$$\Delta_g C(I_\Delta g(x)) = 0 \quad \text{for any } C(x+1) = C(x) \tag{7.9}$$

$$\Delta_g 2^{I_\Delta g(x)} = 2^{I_\Delta g(x)} \tag{7.10}$$

$$\Delta_g \left(I_\Delta g(x)\right)^{\langle n\rangle} = n \left(I_\Delta g(x)\right)^{\langle n-1\rangle} \tag{7.11}$$

$$\Delta_g \operatorname{sind}\left(I_\Delta g(x)\right) = \operatorname{cosd}\left(I_\Delta g(x)\right) \tag{7.12}$$

$$\Delta_g \operatorname{cosd}\left(I_\Delta g(x)\right) = -\operatorname{sind}\left(I_\Delta g(x)\right) \tag{7.13}$$

$$\tag{7.14}$$

$$p(x+1) = g(p(x)) \Rightarrow I_\Delta g(x) = p^{-1}(x) \tag{7.15}$$

$$p(x) = I_\Delta g(x) \Rightarrow f(g(x)) = f(p(p^{-1}x) + k)) \tag{7.16}$$

In the above, 7.15 provides a way of determing $I_\Delta g(x)$, and 7.16 provides a way to convert a functional equation (where all occurences of f are in the form $f(g(x))$ to a difference equation.

7.6. Table

Here, w is an arbitrary constant.

$$g(x) \qquad I_\Delta g(x) \tag{7.17}$$

$$x+h \qquad \frac{x}{h} \tag{7.18}$$

$$ax+b \qquad \log_a\left(\frac{x(a-1)}{b}+1\right) \tag{7.19}$$

$$hx \quad h>0 \qquad \log_h x \tag{7.20}$$

$$-hx \quad h>0 \qquad \frac{\ln x}{\pi i+\ln h} \tag{7.21}$$

$$x^h, \quad h>0 \qquad \log_h \log_w x \tag{7.22}$$

$$x^{-h}, \quad h>0 \qquad \frac{\ln\log_w x}{\pi i+\ln h} \tag{7.23}$$

$$h^x \qquad \operatorname{slog}_h(x) \tag{7.24}$$

$$h-x \qquad \frac{\ln(h-2x)}{\pi i} \tag{7.25}$$

$$\frac{h}{x}, \quad h\neq 0,1 \qquad \frac{\ln\left(\frac{2\ln x}{\ln h}-1\right)}{\pi i} \tag{7.26}$$

8. Trigonometric Equations

$$f(\cos(x)) + f(\sin(x)) = 1 \Rightarrow f(x) = \frac{2}{\pi}\cos^{-1}x + C \tag{8.1}$$

$$f(\cos(x)) + f(\sin(x)) = f(x) \Rightarrow f(x) = 2^{\frac{2}{\pi}\cos^{-1}x}C \tag{8.2}$$

Part II.

Transforms

Here, $\Delta(x)$ is the unit integer pulse function:

$$\Delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases} \tag{8.3}$$

9. z-Transform

This section deals with the one-sided z-transform.

9.1. Properties

If $F(z) = \mathcal{Z}[f(x)]$ and $G(z) = \mathcal{Z}[g(x)]$, then

$$\mathcal{Z}[af(x) + bg(x)] = aF(z) + bG(z) \tag{9.1}$$

$$\mathcal{Z}[f(x-y)] = z^{-y}F(z) \tag{9.2}$$

$$\mathcal{Z}[f(-x)] = F\left(\frac{1}{z}\right) \tag{9.3}$$

$$\mathcal{Z}[a^x f(x)] = F\left(\frac{z}{a}\right) \tag{9.4}$$

$$\mathcal{Z}[\cos(ax)f(x)] = \frac{1}{2}\left[F\left(e^{ai}z\right) + F(e^{-ai}z)\right] \tag{9.5}$$

$$\mathcal{Z}[\sin(ax)f(x)] = \frac{i}{2}\left[F\left(e^{ai}z\right) - F(e^{-ai}z)\right] \tag{9.6}$$

$$\mathcal{Z}[x^k f(x)] = -z\frac{d}{dz}\mathcal{Z}[x^{k-1}f(x)] \tag{9.7}$$

$$\mathcal{Z}[\Delta f(x)] = (z-1)F(z) \tag{9.8}$$

$$\mathcal{Z}\left[\sum_{k=0}^{x} f(k)\right] = \frac{zF(z)}{z-1} \tag{9.9}$$

$$\mathcal{Z}\left[\sum_{k=0}^{x-1} f(k)\right] = \frac{F(z)}{z-1} \tag{9.10}$$

$$\mathcal{Z}[f(x) * g(x)] = F(z)G(z) \tag{9.11}$$

$$\mathcal{Z}[f(x)g(x)] = \frac{1}{2\pi i}\oint_C \frac{F(w)G(\frac{z}{w})}{z}dw \tag{9.12}$$

$$\mathcal{Z}[\mathcal{L}[f(x)]] = \int_0^\infty \frac{f(t)}{1-e^{-t}z^{-1}}dt \tag{9.13}$$

$$\lim_{z\to\infty} F(z) = f(0) \tag{9.14}$$

$$\lim_{x\to\infty} f(x) = \lim_{z\to 1}(1-z^{-1})F(z) \tag{9.15}$$

$$\sum_{x=0}^{\infty} f(x)g^*(x) = \frac{1}{2\pi i}\oint_C \frac{F(w)G(\frac{1}{w^*})}{z}dw \tag{9.16}$$

9.2. Pairs

9.2.1. Basics

$$\Delta(x) \to 1 \tag{9.17}$$

$$\Delta(x-n) \to z^{-n} \qquad z \neq 0 \tag{9.18}$$

$$1 \to \frac{z}{z-1} \tag{9.19}$$

$$u(x) \to \frac{z}{z-1} \tag{9.20}$$

$$u(x-k) \to \frac{z}{z^k(z-1)} \tag{9.21}$$

$$a^x \to \frac{z}{z-a} \tag{9.22}$$

$$\mathcal{F}(x) \to \frac{z}{z^2-z-1} \tag{9.23}$$

$$x \to \frac{z}{(z-1)^2} \tag{9.24}$$

$$x^2 \to \frac{z(z+1)}{(z-1)^3} \tag{9.25}$$

$$x^3 \to \frac{z(z^2+4z+1)}{(z-1)^4} \tag{9.26}$$

$$x^4 \to \frac{z(z^3+11z^2+11z+1)}{(z-1)^6} \tag{9.27}$$

$$x^5 \to \frac{z(z^4+26z^3+66z^2+26z+1)}{(z-1)^6} \tag{9.28}$$

$$x^{\langle n \rangle} \to \frac{n!z}{(z-1)^{n+1}} \tag{9.29}$$

$$xa^x \to \frac{az}{(z-a)^2} \qquad |z| > |a| \tag{9.30}$$

$$x^2a^x \to \frac{az(z+a)}{(z-a)^3} \qquad |z| > |a| \tag{9.31}$$

$$x^3a^x \to \frac{az(z^2+4az+a^2)}{(z-a)^4} \tag{9.32}$$

$$\frac{1}{x+1} \to \ln\left(\frac{z}{z-1}\right) \tag{9.33}$$

$$\frac{a^{x+1}}{x+1} \to \ln\left(\frac{z}{z-a}\right) \tag{9.34}$$

$$\frac{1-a^{x+1}}{x+1} \to \ln\left(\frac{z-a}{z-1}\right) \tag{9.35}$$

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^{\langle n \rangle} \to \sum_{n=0}^{\infty} \frac{a_n z}{(z-1)^{n+1}} \tag{9.36}$$

9.2.2. Trigonometric Functions

$$\sin(bx) \to \frac{z\sin b}{z^2 - 2z\cos b + 1} \tag{9.37}$$

$$\cos(bx) \to \frac{z(z - \cos b)}{z^2 - 2z\cos b + 1} \tag{9.38}$$

$$a^x \sin(bx) \to \frac{az\sin b}{z^2 - 2az\cos b + a^2} \qquad |z| > e^{-a} \tag{9.39}$$

$$a^x \cos(bx) \to \frac{z(z - a\cos b)}{z^2 - 2az\cos b + a^2} \qquad |z| > e^{-a} \tag{9.40}$$

$$\frac{\sin b(x+1)}{x+1} \to b + \operatorname{atan}\frac{\sin b}{z - \cos b} \tag{9.41}$$

$$\frac{\cos b(x+1)}{x+1} \to \ln\frac{z}{\sqrt{z^2 - 2z\cos b + 1}} \tag{9.42}$$

9.2.3. Hyperbolic Functions

$$\sinh(bx) \to \frac{z\sinh b}{z^2 - 2z\cosh b + 1} \qquad |z| > e^{-b} \tag{9.43}$$

$$\cosh(bx) \to \frac{z(z - \cosh b)}{z^2 - 2z\cosh b + 1} \qquad |z| > e^{-b} \tag{9.44}$$

$$a^x \sinh(bx) \to \frac{az\sinh b}{z^2 - 2az\cosh b + a^2} \tag{9.45}$$

$$a^x \cosh(bx) \to \frac{z(z - a\cosh b)}{1 - 2az\cosh b + a^2} \tag{9.46}$$

9.2.4. Discreet Trigonometric Functions

$$\operatorname{sind}(bx) \to \frac{\operatorname{sind}(b)z^{-1}}{1 - 2\operatorname{cosd}(b)z^{-1} + 2^b z^{-2}} \tag{9.47}$$

$$\operatorname{cosd}(bx) \to \frac{1 - \operatorname{cosd}(b)z^{-1}}{1 - 2\operatorname{cosd}(b)z^{-1} + 2^b z^{-2}} \tag{9.48}$$

$$\tag{9.49}$$

9.2.5. Gamma and Related Functions

$$\Gamma(x) \to^? e^{-z}C \tag{9.50}$$
$$\frac{a^x}{\Gamma(x+1)} \to e^{a/z} \tag{9.51}$$
$$\frac{a^{2x+1}}{\Gamma(2x+2)} \to \sinh\left(\frac{c}{z}\right) \tag{9.52}$$
$$\frac{a^{2x}}{\Gamma(2x+1)} \to \cosh\left(\frac{c}{z}\right) \tag{9.53}$$
$$\frac{(\ln a)^x}{\Gamma(x+1)} \to a^{1/z} \tag{9.54}$$
$$\binom{y}{x} a^{y-x} b^x \to \frac{(az+b)^y}{z^y} \tag{9.55}$$
$$\binom{x+y}{y} b^x \to \frac{z^{y+1}}{(z-b)^{y+1}} \tag{9.56}$$
$$G(x) \to^? \frac{e^{-z}C}{z-1} \tag{9.57}$$
$$xG(x) \to^? \frac{e^{-z}z^2C}{(z-1)^2} \tag{9.58}$$
$$\varphi_0(x+1) - \varphi_0(1) \to \frac{z}{z-1} \ln\left(\frac{z}{z-1}\right) \tag{9.59}$$
$$B(x,a) \to^? (z-1)^{a-1}C \tag{9.60}$$
$$\gamma(x,a) \to^? \operatorname{Ei}(a-z)e^z + Ce^z \tag{9.61}$$
$$\Gamma(x,a) \to^? -\operatorname{Ei}(a-z)e^z + Ce^z \tag{9.62}$$

9.2.6. Exponential Sums (?)

$$\sum \frac{\sin b(x+1)}{b(x+1)} \to \frac{1}{2b(z-1)} \ln\left(\frac{z-\cos b-\sin b}{z-\cos b+\sin b}\right) \tag{9.63}$$
$$\sum \frac{\cos b(x+1)}{b(x+1)} \to \frac{1}{2b(z-1)} \ln\left(z^2 - 2\cos bz + 2^b\right) \tag{9.64}$$
$$\operatorname{Bs}(b(x+1)) \to -\frac{\ln(1-2^b z^{-1})}{b(z-1)} \tag{9.65}$$
$$\operatorname{Ss}(b(x+1)) \to \frac{1}{2b(z-1)} \ln\left(\frac{z-\operatorname{cosd}(b)-\operatorname{sind}(b)}{z-\operatorname{cosd}(b)+\operatorname{sind}(b)}\right) \tag{9.66}$$
$$\operatorname{Cs}(b(x+1)) \to \frac{1}{2b(z-1)} \ln\left(z^2 - 2\operatorname{cosd}(b)z + 2^b\right) \tag{9.67}$$

9.2.7. Special Functions

The following functions are used in the table below:

- $J_k(t)$, Bessel polynomials
- $H_k(t)$, Hermite polynomials
- $L_k(t)$, Laguerre polynomials (?)
- $P_k(t)$, Legendre polynomials

- $T_k(t)$, Chebyshev polynomials

$$a^x P_x(t) \to \frac{z}{\sqrt{z^2 + -2taz + a^2}} \tag{9.68}$$

$$a^x P_x^m(t) \to \frac{(2m)! z^{m+1}(1-t^2)^{m/2} a^m}{2^m m! (z^2 - 2atz + a^2)^{m+1/2}} \tag{9.69}$$

$$\frac{P_x(t)}{x!} \to \exp\left(\frac{t}{z}\right) J_0\left(\frac{\sqrt{1-x^2}}{z}\right) \tag{9.70}$$

$$\frac{P_x^m(t)}{(x+m)!} \to (-1)^m \exp\left(\frac{t}{z}\right) J_m\left(\frac{\sqrt{1-x^2}}{z}\right) \tag{9.71}$$

$$a^x T_x(t) \to \frac{z(z-at)}{z^2 - 2atz + a^2} \tag{9.72}$$

$$\frac{L_x(t)}{x!} \to \frac{ze^{-t/(z-1)}}{z-1} \tag{9.73}$$

$$\frac{H_x(t)}{x!} \to e^{t/z - 1/2z^2} \tag{9.74}$$

$$\frac{L_x^m}{x!} \to \frac{(-1)^m z}{(z-1)^{m+1}} \exp\left(\frac{-t}{z-1}\right) \tag{9.75}$$

$$\tag{9.76}$$

9.2.8. Sums

The following functions are used in the table below:

- $J_k(t)$, Bessel polynomials
- $H_k(t)$, Hermite polynomials
- $L_k(t)$, Laguerre polynomials
- $P_k(t)$, Legendre polynomials
- $T_k(t)$, Chebyshev polynomials

$$\sum_{k=0}^{x-1} \frac{1}{\Gamma(k+1)} \to \frac{e^{1/z}}{z-1} \tag{9.77}$$

$$\sum_{k=0}^{x} \frac{a^k b^{x-k}}{k!} \to \frac{e^{a/z} z}{z-b} \qquad b^2 < 1 \tag{9.78}$$

$$\sum_{k=0}^{x} a^k b^{x-k} J_k(t) \to \frac{z}{z-b} \exp\left(\frac{t(a^2+z^2)}{2az}\right) \qquad b^2 < 1; d \in \mathbb{R} \tag{9.79}$$

$$\sum_{k=0}^{x} \frac{a^k b^{x-k}}{k!} H_k(t) \to \frac{z}{z-b} \exp\left(-\frac{dz(2tz+a)}{2z^2}\right) \qquad b^2 < 1 \tag{9.80}$$

$$\sum_{k=0}^{x} \frac{a^k b^{x-k}}{k!} \frac{d}{dt} L_k(t) \to \frac{(-a)^m z^2}{(z-a)^{m+1}(z-b)} \exp\left(\frac{at}{1-z}\right) \qquad a^2 < 1; b^2 < 1 \tag{9.81}$$

$$\sum_{k=0}^{x} \frac{b^{x-k}\left[(-a)^k - (-c)^k\right]}{k} \to \frac{z}{z-b} \ln\left(\frac{z+c}{z+a}\right) \qquad a^2 < 1; b^2 < 1; c^2 < 1 \tag{9.82}$$

$$\sum_{k=0}^{x} a^k b^{x-k} \frac{d}{dt} P_k(t) \to \frac{(2m)!a^m}{2^m m!} \frac{z^{m+2}}{(z-b)(z^2-2atz+a^2)^{m+1/2}} \qquad a^2 < 1; b^2 < 1 \tag{9.83}$$

$$\sum_{k=0}^{x} a^k b^{x-k} T_k(t) \to \frac{z^2(z-at)}{(z-bz)(z^2-2atz+a)} \qquad a^2 < 1; b^2 < 1 \tag{9.84}$$

10. Binomial Transforms

10.1. Definition

The binomial transform $F(k) = \mathcal{B}[f(x)]$ of a function $f(x)$ is defined by:

$$F(k) = \sum_{x=0}^{k} (-1)^{k-x} \binom{k}{x} f(x) \tag{10.1}$$

The inverse transform $f(x) = \mathcal{B}^{-1}[F(x)]$ of a function $f(x)$ is defined by:

$$f(x) = \sum_{k=0}^{x} \binom{x}{k} F(k) \tag{10.2}$$

10.2. Properties

Here, E is the shift operator: $E_x f(x) = f(x-1)$.

$$\mathcal{B}[af(x) + bg(x)] = a\mathcal{B}[f(x) + b\mathcal{B}[g(x)] \tag{10.3}$$

$$\mathcal{B}^{-1}[E_k f(k)] = f(-1) + xE_x\mathcal{B}^{-1}\left[\frac{f(k)}{k+1}\right] \tag{10.4}$$

$$\mathcal{B}^{-1}[kE_k f(k)] = xE_x\mathcal{B}^{-1}[f(k)] \tag{10.5}$$

$$\mathcal{B}^{-1}\left[\sum_{m=0}^{\infty} a_m k^{\langle m \rangle}\right] = 2^x \sum_{m=0}^{\infty} a_m 2^{-m} x^{\langle m \rangle} \tag{10.6}$$

$$f(x) = \sum_{k=0}^{\infty} F(k) \frac{x^{\langle k \rangle}}{k!} \tag{10.7}$$

$$F(k) = \Delta^k f(x)|_{x=0} \tag{10.8}$$

10.3. Pairs

10.3.1. Basics

$$1 \to \Delta(k) \tag{10.9}$$

$$x \to \Delta(k-1) \tag{10.10}$$

$$x^{\langle n \rangle} \to n!\Delta(k-n) \tag{10.11}$$

$$x! \to !k \tag{10.12}$$

$$2^x \to 1 \tag{10.13}$$

$$a^x \to (a-1)^k \tag{10.14}$$

$$\frac{1}{2^x} \to (-1)^k \frac{1}{2^k} \tag{10.15}$$

10.4. Pairs (Inverse)

10.4.1. Basics

$$1 \to 2^x \tag{10.16}$$

$$k \to x2^{x-1} \tag{10.17}$$

$$k^{\langle n \rangle} \to x^{\langle n \rangle} 2^{x-n} \qquad n \geq 0 \tag{10.18}$$

$$k^{\langle -1 \rangle} \to \frac{1}{x+1}(2^{x+1} - 1) \tag{10.19}$$

$$k^{\langle -2 \rangle} \to x^{\langle -2 \rangle}(2^{x+2} - x - 3) \tag{10.20}$$

$$k^{\langle -3 \rangle} \to x^{\langle -3 \rangle}(2^{x+3} - \frac{x^2 + 9x + 14}{2}) \tag{10.21}$$

$$k^{\langle -n \rangle} \to x^{\langle -2 \rangle} \left(2^{x+n} - \sum_{k=0}^{n-1} \binom{x+n}{k} \right) \tag{10.22}$$

$$a^k \to (a+1)^x \tag{10.23}$$

10.4.2. Discrete Trigonometric

$$\operatorname{sind} k \to 2^x \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} 2^{-2m-1} x^{\langle 2m+1 \rangle} \tag{10.24}$$

$$\operatorname{cosd} k \to 2^x \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} 2^{-2m} x^{\langle 2m \rangle} \tag{10.25}$$

$$\tag{10.26}$$

10.4.3. Gamma and Related Functions

$$\varphi_0(k+1) \to \gamma 2^x + 2^x \sum_{m=1}^{\infty} (-1)^m (m-1)! 2^{-m} x^{\langle m \rangle} \tag{10.27}$$

$$\Gamma(k+1) \to \sum_{k=0}^{x} x^{\langle k \rangle} \tag{10.28}$$

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